

Dynamical Barriers in the Dyson Hierarchical model via Real Space Renormalization

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The Dyson hierarchical one-dimensional Ising model of parameter $\sigma > 0$ contains long-ranged ferromagnetic couplings decaying as $1/r^{1+\sigma}$ in terms of the distance r . We study the stochastic dynamics near zero-temperature via the Real Space Renormalization introduced in our previous work (C. Monthus and T. Garel, arXiv:1212.0643) in order to compute explicitly the equilibrium time $t_{eq}(L)$ as a function of the system size L . For $\sigma < 1$ where the static critical temperature for the ferromagnetic transition is finite $T_c > 0$, we obtain that dynamical barriers grow as the power-law: $\ln t_{eq}(L) \simeq \beta \left(\frac{4J_0}{3(2^{1-\sigma}-1)} \right) L^{1-\sigma}$. For $\sigma = 1$ where the static critical temperature vanishes $T_c = 0$, we obtain that dynamical barriers grow logarithmically as : $\ln t_{eq}(L) \simeq [\beta \left(\frac{4J_0}{3 \ln 2} \right) - 1] \ln L$. We also compute finite contributions to the dynamical barriers that can depend on the choice of transition rates satisfying detailed balance.

I. INTRODUCTION

The Dyson hierarchical Ising model [1] has been introduced as a ferromagnetic model where the partition function could be analyzed via exact renormalization. The hierarchical couplings are chosen to mimic effective long-range power-law couplings $J(r) \simeq 1/r^{1+\sigma}$ in one dimension so that a phase transition with finite T_c is possible in the region $0 < \sigma < 1$. This type of hierarchical model has thus attracted a great interest in statistical physics, both among mathematicians [2–5] and among physicists [6–9]. Note that Dyson hierarchical models have been also introduced in the field of quenched disordered models, in particular for random fields [10, 11], spin-glasses [12–14], and for Anderson localization [15–22].

In the present paper, we consider the Dyson hierarchical ferromagnetic Ising model and we focus on the stochastic dynamics satisfying detailed-balance, such as the Glauber dynamics [23]. We do not consider the coarsening dynamics starting from a random initial condition (see the review [24]) but focus instead on the equilibrium time t_{eq} near zero temperature, i.e. the time needed to go from one ground state (where all spins take the value $+1$) to the opposite ground state (where all spins take the value -1). In a previous work, we have introduced a real-space renormalization procedure to determine this equilibrium time t_{eq} [25] as a function of the system size. Here we solve the corresponding RG flow for the Dyson hierarchical Ising model.

The paper is organized as follows. In section II, we recall the important properties of the equilibrium of the Dyson hierarchical Ising model. In section III, we study the stochastic dynamics satisfying detailed-balance via the real space renormalization procedure introduced in [25]. The explicit solutions of the renormalization flow for the 'simple' dynamics and for the Glauber dynamics are presented in sections IV and V respectively. Our conclusions are summarized in section VI. Finally in Appendix A, we discuss the link between dynamical barriers and the energy-cost of a single domain wall.

II. REMINDER ON THE STATICS OF THE DYSON HIERARCHICAL ISING MODEL

A. Definition of the Dyson hierarchical Ising model

The Dyson hierarchical Ising model is a model of 2^N classical spins $S_i = \pm 1$ where each configuration $\mathcal{C} = (S_1, S_2, \dots, S_{2^N})$ has for energy

$$\begin{aligned}
 U_{2^N}(S_1, \dots, S_{2^N}) = & - \sum_{i < j} J_{ij} S_i S_j = -J_0 [S_1 S_2 + S_3 S_4 + S_5 S_6 + S_7 S_8 + \dots] \\
 & - J_1 \left[\left(\frac{S_1 + S_2}{2} \right) \left(\frac{S_3 + S_4}{2} \right) + \left(\frac{S_5 + S_6}{2} \right) \left(\frac{S_7 + S_8}{2} \right) + \dots \right] \\
 & - J_2 \left[\left(\frac{S_1 + S_2 + S_3 + S_4}{2^2} \right) \left(\frac{S_5 + S_6 + S_7 + S_8}{2^2} \right) + \dots \right] - \dots \\
 & - J_{N-1} \left(\frac{S_1 + \dots + S_{2^{N-1}}}{2^{N-1}} \right) \left(\frac{S_{2^{N-1}+1} + \dots + S_{2^N}}{2^{N-1}} \right)
 \end{aligned} \tag{1}$$

where the positive couplings J_n depend on n via

$$J_n = J_0 2^{(1-\sigma)n} \quad (2)$$

To make the link with the physics of long-range one-dimensional models, it is convenient to consider that the sites i of the Dyson model are displayed on a one-dimensional lattice, with a lattice spacing unity. Then the site $i = 1$ is coupled via the coupling $J_n/(2^n)^2$ to each spin of index $2^{n-1} < i \leq 2^n$. At the scaling level, the hierarchical Dyson model is thus somewhat equivalent to the following power-law dependence in the real-space distance $L_n = 2^n$

$$J^{eff}(L_n) = \frac{J_n}{(2^n)^2} = \frac{J_0}{2^{n(1+\sigma)}} = \frac{J_0}{L_n^{1+\sigma}} \quad (3)$$

The parameter σ is thus the important parameter of the model.

The ground state energy where all spins have the same sign

$$\begin{aligned} U_{2^N}^{GS} = U_{2^N}(S_1 = 1, \dots, S_{2^N} = 1) &= -J_0 2^{N-1} - J_1 2^{N-2} - J_2 2^{N-3} - \dots - J_{N-1} \\ &= -J_0 2^{N-1} \frac{1 - 2^{-\sigma N}}{1 - 2^{-\sigma}} = -\frac{J_0}{2} L_N \frac{1 - L_N^{-\sigma}}{1 - 2^{-\sigma}} \end{aligned} \quad (4)$$

is extensive in the number $L_N = 2^N$ of spins in the region

$$\sigma > 0 \quad (5)$$

The energy cost of the configuration where the first 2^{N-1} spins are (-1) , whereas the other 2^{N-1} spins are $(+1)$

$$U_{2^N}^{(2^{N-1}, 2^{N-1})} - U_{2^N}^{GS} = 2J_{N-1} = 2J_0 2^{(1-\sigma)(N-1)} = J_0 2^\sigma L_N^{1-\sigma} \quad (6)$$

grows with the distance $L_N = 2^N$ for $\sigma < 1$, remains constant for $\sigma = 1$, and decays for $\sigma > 1$. It has been shown [1] that the critical temperature T_c for the ferromagnetic transition is finite for $\sigma < 1$ only

$$\begin{aligned} T_c(\sigma < 1) &> 0 \\ T_c(\sigma \geq 1) &= 0 \end{aligned} \quad (7)$$

For $\sigma > 1$, note that Eq 6 is not very physical since the energy cost of a single domain wall in the middle of the sample becomes exponentially small instead of remaining finite in the usual one-dimensional ferromagnetic chain. So, in the following, we will consider only the region

$$0 < \sigma \leq 1 \quad (8)$$

B. Real-space renormalization for the equilibrium near zero-temperature

In this article, we will focus on the regime 'near zero temperature'

$$T \ll J_0 \quad (9)$$

where the partition function is dominated by the two ferromagnetic ground states. Then the real-space renormalization of the partition function becomes very simple. The consecutive spins (S_{2i-1}, S_{2i}) that are coupled by J_0 are grouped into a single renormalized spin $S_i^R = \frac{S_{2i-1} + S_{2i}}{2}$ corresponding to the ferromagnetic cluster of the two spins

$$\begin{aligned} |S_i^R = 1\rangle &\equiv |S_{2i-1} = 1\rangle |S_{2i} = 1\rangle \\ |S_i^R = -1\rangle &\equiv |S_{2i-1} = -1\rangle |S_{2i} = -1\rangle \end{aligned} \quad (10)$$

In terms of these 2^{N-1} renormalized spins for $i = 1, 2, \dots, 2^{N-1}$, the energy of Eq. 1 becomes

$$\begin{aligned} U_{2^N}(S_1, \dots, S_{2^N}) &= -J_0 2^{N-1} - J_1 [S_1^R S_2^R + S_3^R S_4^R + \dots] - J_2 \left[\left(\frac{S_1^R + S_2^R}{2} \right) \left(\frac{S_3^R + S_4^R}{2} \right) + \dots \right] - \dots \\ &\quad - J_{N-1} \left(\frac{S_1^R + \dots + S_{2^{N-2}}^R}{2^{N-2}} \right) \left(\frac{S_{2^{N-2}+1}^R + \dots + S_{2^{N-1}}^R}{2^{N-2}} \right) \\ &= -J_0 2^{N-1} + 2^{1-\sigma} U_{2^{N-1}}(S_1^R, \dots, S_{2^{N-1}}^R) \end{aligned} \quad (11)$$

So if one considers the partition function where the ratio U/T enters, one obtains that Eq. 11 corresponds to the following renormalization of the temperature upon the elimination of the lowest generation

$$T_R = 2^{\sigma-1} T \quad (12)$$

This is in agreement with Eq. 7 : for $\sigma < 1$, T_R flows towards the attractive fixed point $T = 0$; for $\sigma > 1$, T_R flows away from the unstable fixed point $T_c = 0$.

To prepare the following sections, it is also convenient to describe the renormalization of the local field. In the initial model of Eq. 1, the local field on spin S_i is defined as

$$B_i = -\frac{\partial U_{2^N}(S_1, \dots, S_{2^N})}{\partial S_i} = \sum_{j \neq i} J_{ij} S_j \quad (13)$$

i.e. for instance for the two first spins $i = 1, 2$

$$\begin{aligned} B_1 &= J_0 S_2 + \frac{J_1}{2} \left(\frac{S_3 + S_4}{2} \right) + \frac{J_2}{2^2} \left(\frac{S_5 + S_6 + S_7 + S_8}{2^2} \right) + \dots + \frac{J_{N-1}}{2^{N-1}} \left(\frac{S_{2^{N-1}+1} + \dots + S_{2^N}}{2^{N-1}} \right) \\ B_2 &= J_0 S_1 + \frac{J_1}{2} \left(\frac{S_3 + S_4}{2} \right) + \frac{J_2}{2^2} \left(\frac{S_5 + S_6 + S_7 + S_8}{2^2} \right) + \dots + \frac{J_{N-1}}{2^{N-1}} \left(\frac{S_{2^{N-1}+1} + \dots + S_{2^N}}{2^{N-1}} \right) \end{aligned} \quad (14)$$

After the renormalization where (S_1, S_2) have been grouped into the renormalized spin $S_1^R = (S_1 + S_2)/2$ (Eq 10), the renormalized local field B_1^R on S_1^R reads

$$\begin{aligned} B_1^R &= J_1 \left(\frac{S_3 + S_4}{2} \right) + \frac{J_2}{2} \left(\frac{S_5 + S_6 + S_7 + S_8}{2^2} \right) + \dots + \frac{J_{N-1}}{2^{N-2}} \left(\frac{S_{2^{N-1}+1} + \dots + S_{2^N}}{2^{N-1}} \right) \\ &= 2^{1-\sigma} J_0 \left[S_2^R + 2^{-\sigma} \left(\frac{S_3^R + S_4^R}{2} \right) + \dots + 2^{-\sigma(N-2)} \left(\frac{S_{2^{N-2}+1}^R + \dots + S_{2^{N-1}}^R}{2^{N-2}} \right) \right] \end{aligned} \quad (15)$$

i.e. it is renormalized by the same factor $2^{1-\sigma}$ as the couplings as it should.

III. REAL SPACE RENORMALIZATION FOR THE DYNAMICS OF THE DYSON MODEL

A. Stochastic single-spin-flip dynamics satisfying detailed-balance

We consider the stochastic dynamics generated by the Master Equation for the probability $P_t(\mathcal{C})$ to be in configuration \mathcal{C} at time t

$$\frac{dP_t(\mathcal{C})}{dt} = \sum_{\mathcal{C}'} P_t(\mathcal{C}') W(\mathcal{C}' \rightarrow \mathcal{C}) - P_t(\mathcal{C}) \sum_{\mathcal{C}'} W(\mathcal{C} \rightarrow \mathcal{C}') \quad (16)$$

where the transition rates $W(\mathcal{C} \rightarrow \mathcal{C}')$ satisfy the detailed balance condition

$$e^{-\beta U(\mathcal{C})} W(\mathcal{C} \rightarrow \mathcal{C}') = e^{-\beta U(\mathcal{C}')} W(\mathcal{C}' \rightarrow \mathcal{C}) \quad (17)$$

For the Ising model of Eq 1, where the energy is of the form

$$U(\mathcal{C}) = - \sum_{i < j} J_{ij} S_i S_j \quad (18)$$

with the local fields (Eq 13)

$$B_k = -\frac{\partial U(\mathcal{C})}{\partial S_k} = \sum_{i \neq k} J_{ki} S_i \quad (19)$$

it is natural to consider the following single-spin-flip transition rates for the flip of the spin S_k

$$W(S_k \rightarrow -S_k) = G_0(B_k) e^{-\beta S_k B_k} \quad (20)$$

where the function $G_0(B)$ is an even positive function of B (see [25] for more details)

$$G_0(B) = G_0(-B) > 0 \quad (21)$$

In the following, we will consider as two important examples the 'simple' dynamics

$$\text{Simple Dynamics : } G_0^{\text{simple}}(B) = 1 \quad (22)$$

and the Glauber dynamics [23]

$$\text{Glauber Dynamics : } G_0^{\text{Glauber}}(B) = \frac{1}{2 \cosh \beta B} \quad (23)$$

B. Mapping onto a quantum Hamiltonian

As recalled in detail in [25], the Master equation with the transition rates of Eq. 20 can be mapped via a similarity transformation onto the quantum Hamiltonian [26–30]

$$H = \sum_{k=1}^{2^N} G_0 \left(\sum_{i \neq k} J_{ik} \sigma_i^z \right) \left(e^{-\beta \sigma_k^z (\sum_{i \neq k} J_{ik} \sigma_i^z)} - \sigma_k^x \right) \quad (24)$$

for 2^N quantum spins described by Pauli matrices, with the following properties. The ground state energy exactly vanishes $E_0 = 0$ and represents the thermal equilibrium. The smallest non-vanishing energy $E_1(2^N)$ determines the equilibrium time $t_{eq}(2^N)$ (defined as the largest relaxation time of the Master Eq. 16)

$$t_{eq}(2^N) = \frac{1}{E_1(2^N)} \quad (25)$$

C. Real Space Renormalization of the associated quantum Hamiltonian

In our previous work [25], we have introduced a real-space renormalization procedure for quantum Hamiltonian of the form of Eq. 24 to determine E_1 near zero-temperature. We now describe the application to the Dyson hierarchical model. The quantum Hamiltonian associated to the stochastic dynamics of the Dyson hierarchical Ising model can be rewritten as a sum of elementary operators (Eq 24)

$$H = \sum_{i=1}^{2^{N-1}} (h_{2i-1} + h_{2i}) \quad (26)$$

$$h_{2i-1} + h_{2i} = G_0 \left(J_0 \sigma_{2i}^z + \frac{B_i^R}{2} \right) \left(e^{-\beta \sigma_{2i-1}^z \left(J_0 \sigma_{2i}^z + \frac{B_i^R}{2} \right)} - \sigma_{2i-1}^x \right) + G_0 \left(J_0 \sigma_{2i-1}^z + \frac{B_i^R}{2} \right) \left(e^{-\beta \sigma_{2i}^z \left(J_0 \sigma_{2i-1}^z + \frac{B_i^R}{2} \right)} - \sigma_{2i}^x \right)$$

with B_i^R is the static renormalized local field introduced Eq. 15.

Let us consider the first two spins : as explained in [25], the sum

$$h_1 + h_2 = G_0 \left(J_0 \sigma_2^z + \frac{B_1^R}{2} \right) \left(e^{-\beta \sigma_1^z \left(J_0 \sigma_2^z + \frac{B_1^R}{2} \right)} - \sigma_1^x \right) + G_0 \left(J_0 \sigma_1^z + \frac{B_1^R}{2} \right) \left(e^{-\beta \sigma_2^z \left(J_0 \sigma_1^z + \frac{B_1^R}{2} \right)} - \sigma_2^x \right) \quad (27)$$

can be renormalized onto the following operator describing the flip of the renormalized spin $S^R = (S_1 + S_2)/2$ (Eq 10)

$$h_{(1,2)}^R \equiv G_1^R(B_1^R) \left(e^{-\beta \sigma_R^z B_1^R} - \sigma_R^x \right) \quad (28)$$

where the renormalized amplitude $G_1^R(B)$ can be computed from the function B_0 via

$$G_1^R(B) = e^{-\beta J_0} \frac{2G_0(J_0 + \frac{B}{2})G_0(J_0 - \frac{B}{2})}{e^{\beta \frac{B}{2}} G_0(J_0 + \frac{B}{2}) + e^{-\beta \frac{B}{2}} G_0(J_0 - \frac{B}{2})} \quad (29)$$

or equivalently using the inverse

$$\frac{1}{G_1^R(B)} = \frac{e^{\beta J_0}}{2} \left[\frac{e^{\beta \frac{B}{2}}}{G_0(J_0 - \frac{B}{2})} + \frac{e^{-\beta \frac{B}{2}}}{G_0(J_0 + \frac{B}{2})} \right] \quad (30)$$

In conclusion, the real space renormalization of the quantum Hamiltonian corresponds for the Dyson model to the renormalization of the function $G(B)$ starting from the initial condition $G_0(B)$ that defines the initial dynamics (Eq 21).

D. Equilibrium time $t_{eq}(2^N)$ for a finite system of 2^N spins

We apply iteratively the previous renormalization rule as follows :

(0) The initial dynamics concerns the Dyson hierarchical model of 2^N spins with the couplings (J_0, \dots, J_{N-1}) and the function $G_0(B)$ that defines the transition rates of Eq 20.

(1) After 1 RG step, we have a Dyson hierarchical model of 2^{N-1} renormalized spins with the couplings (J_1, \dots, J_{N-1}) and the function $G_1^R(B)$ obtained by Eq 30

$$\frac{1}{G_1^R(B)} = \frac{e^{\beta J_0}}{2} \sum_{\epsilon_1=\pm} \frac{e^{-\beta \epsilon_1 \frac{B}{2}}}{G_0(J_0 + \epsilon_1 \frac{B}{2})} \quad (31)$$

(2) After 2 RG steps, we have a Dyson hierarchical model of 2^{N-2} renormalized spins with the couplings (J_2, \dots, J_{N-1}) and the function $G_2^R(B)$ obtained by Eq 30

$$\frac{1}{G_2^R(B)} = \frac{e^{\beta J_1}}{2} \sum_{\epsilon_2=\pm} \frac{e^{-\beta \epsilon_2 \frac{B}{2}}}{G_1(J_1 + \epsilon_2 \frac{B}{2})} \quad (32)$$

...

(k) After k RG steps, we have a Dyson hierarchical model of 2^{N-k} renormalized spins with the couplings (J_k, \dots, J_{N-1}) and the function $G_k^R(B)$

$$\frac{1}{G_k^R(B)} = \frac{e^{\beta J_{k-1}}}{2} \sum_{\epsilon_k=\pm} \frac{e^{-\beta \epsilon_k \frac{B}{2}}}{G_{k-1}(J_{k-1} + \epsilon_k \frac{B}{2})} \quad (33)$$

..

(N) After N RG steps, only one spin remains, whose flipping is governed by the function $G_N^R(B)$

$$\frac{1}{G_N^R(B)} = \frac{e^{\beta J_{N-1}}}{2} \sum_{\epsilon_N=\pm} \frac{e^{-\beta \epsilon_N \frac{B}{2}}}{G_{N-1}(J_{N-1} + \epsilon_N \frac{B}{2})} \quad (34)$$

However since this spin is alone, there is no local field $B = 0$, so we just have to compute the final number $G_N^{final} = G_N^R(B = 0)$

$$\begin{aligned} \frac{1}{G_N^{final}} &= \frac{1}{G_N^R(B=0)} = \frac{e^{\beta J_{N-1}}}{G_{N-1}(J_{N-1})} \\ &= e^{\beta J_{N-1}} \frac{e^{\beta J_{N-2}}}{2} \sum_{\epsilon_{N-1}=\pm} \frac{e^{-\beta \epsilon_{N-1} \frac{J_{N-1}}{2}}}{G_{N-2}(J_{N-2} + \epsilon_{N-1} \frac{J_{N-1}}{2})} \\ &= e^{\beta J_{N-1}} \frac{e^{\beta J_{N-2}}}{2} \frac{e^{\beta J_{N-3}}}{2} \sum_{\epsilon_{N-1}=\pm} \sum_{\epsilon_{N-2}=\pm} \frac{e^{-\beta \epsilon_{N-1} \frac{J_{N-1}}{2}} e^{-\beta \epsilon_{N-2} \frac{(J_{N-2} + \epsilon_{N-1} \frac{J_{N-1}}{2})}{2}}}{G_{N-3} \left[J_{N-3} + \epsilon_{N-2} \frac{(J_{N-2} + \epsilon_{N-1} \frac{J_{N-1}}{2})}{2} \right]} \\ &= \frac{e^{\beta \sum_{n=0}^{N-1} J_n}}{2^{N-1}} \sum_{\epsilon_{N-1}=\pm} \sum_{\epsilon_{N-2}=\pm} \dots \sum_{\epsilon_1=\pm} \frac{e^{-\beta \sum_{n=1}^{N-1} \epsilon_n \frac{B_n}{2}}}{G_0[B_0]} \end{aligned} \quad (35)$$

in terms of the variables \mathcal{B}_n that can be computed from the recurrence

$$\mathcal{B}_{n-1} = J_{n-1} + \epsilon_n \frac{\mathcal{B}_n}{2} \quad (36)$$

with the initial condition

$$\mathcal{B}_N = 0 \quad (37)$$

The first terms read

$$\begin{aligned} \mathcal{B}_{N-1} &= J_{N-1} \\ \mathcal{B}_{N-2} &= J_{N-2} + J_{N-1} \left(\frac{\epsilon_{N-1}}{2} \right) \end{aligned} \quad (38)$$

and more generally for $0 \leq n \leq N-1$, one obtains

$$\mathcal{B}_n = J_n + \sum_{m=n+1}^{N-1} J_m \prod_{k=n+1}^m \left(\frac{\epsilon_k}{2} \right) \quad (39)$$

In the following, we will need

$$\mathcal{B}_0(\epsilon_1, \dots, \epsilon_{N-1}) = J_0 + \sum_{m=1}^{N-1} J_m \prod_{k=1}^m \left(\frac{\epsilon_k}{2} \right) \quad (40)$$

and the sum present in the exponential of Eq. 35

$$\begin{aligned} \Sigma(\epsilon_1, \dots, \epsilon_{N-1}) &\equiv \sum_{n=1}^{N-1} \epsilon_n \frac{\mathcal{B}_n}{2} = \sum_{n=1}^{N-1} \frac{\epsilon_n}{2} \left[J_n + \sum_{m=n+1}^{N-1} J_m \prod_{k=n+1}^m \left(\frac{\epsilon_k}{2} \right) \right] \\ &= \sum_{m=1}^{N-1} J_m \left[\sum_{n=1}^m \prod_{k=n}^m \left(\frac{\epsilon_k}{2} \right) \right] \end{aligned} \quad (41)$$

In conclusion, the equilibrium time $t_{eq}(2^N)$ that can be computed from the final amplitude G_N^{final} [25]

$$t_{eq}(2^N) = \frac{1}{2G_N^{final}} \quad (42)$$

reads

$$t_{eq}(2^N) = \frac{e^{\beta \sum_{n=0}^{N-1} J_n}}{2^N} \sum_{\epsilon_{N-1}=\pm} \sum_{\epsilon_{N-2}=\pm} \dots \sum_{\epsilon_1=\pm} \frac{e^{-\beta \Sigma(\epsilon_1, \dots, \epsilon_{N-1})}}{G_0(\mathcal{B}_0(\epsilon_1, \dots, \epsilon_{N-1}))} \quad (43)$$

where $\mathcal{B}_0(\epsilon_1, \dots, \epsilon_{N-1})$ and $\Sigma(\epsilon_1, \dots, \epsilon_{N-1})$ are given in Eqs 40 and 41. To determine the leading behavior near zero temperature, we have to distinguish whether the initial function $G_0(B)$ does not depend on β (as in the simple dynamics of Eq 22) or depends on β (as in the Glauber dynamics of Eq 23). These two cases are studied respectively in the two following sections.

IV. EQUILIBRIUM TIME FOR THE 'SIMPLE' DYNAMICS

A. Real Space renormalization Solution

For the initial condition of Eq. 22 that defines the 'simple' dynamics, the result of Eq 43 for the equilibrium time becomes

$$t_{eq}^{simple}(2^N) = \frac{e^{\beta \sum_{n=0}^{N-1} J_n}}{2^N} \sum_{\epsilon_{N-1}=\pm} \sum_{\epsilon_{N-2}=\pm} \dots \sum_{\epsilon_1=\pm} e^{-\beta \Sigma(\epsilon_1, \dots, \epsilon_{N-1})} \quad (44)$$

where

$$\begin{aligned}
\Sigma(\epsilon_1, \dots, \epsilon_{N-1}) &= \sum_{m=1}^{N-1} J_m \left[\sum_{n=1}^m \prod_{k=n}^m \left(\frac{\epsilon_k}{2} \right) \right] \\
&= J_1 \frac{\epsilon_1}{2} + J_2 \frac{\epsilon_2}{2} \left[1 + \frac{\epsilon_1}{2} \right] + J_3 \frac{\epsilon_3}{2} \left[1 + \frac{\epsilon_2}{2} + \frac{\epsilon_1 \epsilon_2}{2^2} \right] + \dots \\
&+ J_{N-2} \frac{\epsilon_{N-2}}{2} \left[1 + \frac{\epsilon_{N-3}}{2} + \frac{\epsilon_{N-3} \epsilon_{N-4}}{2^2} + \dots + \frac{\epsilon_{N-3} \epsilon_{N-4} \dots \epsilon_1}{2^{N-3}} \right] \\
&+ J_{N-1} \frac{\epsilon_{N-1}}{2} \left[1 + \frac{\epsilon_{N-2}}{2} + \frac{\epsilon_{N-2} \epsilon_{N-3}}{2^2} + \dots + \frac{\epsilon_{N-2} \epsilon_{N-3} \dots \epsilon_1}{2^{N-2}} \right]
\end{aligned} \tag{45}$$

B. Leading term near zero temperature

At low temperature, the sum of Eq 44 will be dominated by the minimal value of $\Sigma(\epsilon_1, \dots, \epsilon_{N-1})$ of Eq. 41. Since ϵ_{N-1} appears only in the last line, and since the term between the brackets cannot change sign with respect to the first term 1, we are led to the choice

$$\epsilon_{N-1} = -1 \tag{46}$$

and we have now to minimize

$$\begin{aligned}
\Sigma(\epsilon_1, \dots, \epsilon_{N-2}, \epsilon_{N-1} = -1) &= J_1 \frac{\epsilon_1}{2} + J_2 \frac{\epsilon_2}{2} \left[1 + \frac{\epsilon_1}{2} \right] + J_3 \frac{\epsilon_3}{2} \left[1 + \frac{\epsilon_2}{2} + \frac{\epsilon_1 \epsilon_2}{2^2} \right] + \dots \\
&+ \frac{\epsilon_{N-2}}{2} \left(J_{N-2} - \frac{J_{N-1}}{2} \right) \left[1 + \frac{\epsilon_{N-3}}{2} + \frac{\epsilon_{N-3} \epsilon_{N-4}}{2^2} + \dots + \frac{\epsilon_{N-3} \epsilon_{N-4} \dots \epsilon_1}{2^{N-3}} \right] \\
&- \frac{J_{N-1}}{2}
\end{aligned} \tag{47}$$

Since from Eq 2

$$J_{N-2} - \frac{J_{N-1}}{2} = J_0 2^{(1-\sigma)(N-2)} (1 - 2^{-\sigma}) > 0 \tag{48}$$

we are led to the choice

$$\epsilon_{N-2} = -1 \tag{49}$$

and so on, so that the minimum is achieved when all $\epsilon_i = -1$. The corresponding minimum reads using Eq. 2

$$\begin{aligned}
\Sigma_{min} &= \Sigma(\epsilon_1 = -1, \epsilon_2 = -1, \dots, \epsilon_{N-1} = -1) = \sum_{m=1}^{N-1} J_m \left[\sum_{n=1}^m \left(-\frac{1}{2} \right)^{m-n+1} \right] \\
&= -\frac{1}{3} \sum_{m=1}^{N-1} J_m \left[1 - \left(-\frac{1}{2} \right)^m \right]
\end{aligned} \tag{50}$$

so that finally

$$t_{eq}^{simple}(2^N) \simeq \frac{e^{\beta [\sum_{n=0}^{N-1} J_n - \Sigma_{min}]}}{2^N} = \frac{1}{2^N} e^{\beta \left[J_0 + \sum_{m=1}^{N-1} J_m \left(\frac{4 - (-\frac{1}{2})^m}{3} \right) \right]} \tag{51}$$

We may now use Eq 2 to compute explicitly

$$\begin{aligned}
\ln [2^N t_{eq}^{simple}(2^N)] &= \beta \left[J_0 + \sum_{m=1}^{N-1} J_0 2^{(1-\sigma)m} \left(\frac{4 - (-\frac{1}{2})^m}{3} \right) \right] \\
&= \beta J_0 \left[1 + \frac{4}{3} 2^{1-\sigma} \frac{1 - 2^{(1-\sigma)(N-1)}}{1 - 2^{1-\sigma}} + \frac{1}{3} 2^{-\sigma} \frac{1 - (-2^{-\sigma})^{(N-1)}}{1 + 2^{-\sigma}} \right]
\end{aligned} \tag{52}$$

This expression is valid for finite N and we may check the first terms

$$\begin{aligned}\ln [2t_{eq}^{simple}(2)] &= \beta J_0 \\ \ln [2^2 t_{eq}^{simple}(2^2)] &= \beta J_0 [1 + 3 \cdot 2^{-\sigma}] = \beta \left[J_0 + \frac{3}{2} J_1 \right]\end{aligned}\quad (53)$$

To obtain the leading behavior for large N , we should now specify the value of σ .

C. Case $\sigma < 1$: Power-law barrier

For $\sigma < 1$, Eq. 52 becomes in terms of the length $L_N = 2^N$

$$\sigma < 1 \quad : \quad \ln [2^N t_{eq}^{simple}(L_N = 2^N)] = \beta J_0 \left[\frac{4}{3} \frac{L_N^{(1-\sigma)}}{2^{1-\sigma} - 1} + 1 + \frac{4}{3} \frac{1}{2^{\sigma-1} - 1} + \frac{1}{3} \frac{1}{2^\sigma + 1} \right] + O(L_N^{-\sigma}) \quad (54)$$

i.e. the energy barrier near zero temperature scales as the following power-law of the length $L_N = 2^N$

$$\frac{\ln [2^N t_{eq}^{simple}(L_N = 2^N)]}{\beta} \simeq \frac{4J_0}{3(2^{1-\sigma} - 1)} L_N^{(1-\sigma)} \quad (55)$$

D. Case $\sigma = 1$: Logarithmic barrier

For $\sigma = 1$, Eq. 52 becomes in terms of the length $L_N = 2^N$

$$\begin{aligned}\sigma = 1 \quad : \quad \ln [2^N t_{eq}^{simple}(L_N = 2^N)] &= \beta J_0 \left[\frac{4}{3} N - \frac{2}{9} \right] + O(L_N^{-1}) \\ &= \beta J_0 \left[\frac{4}{3 \ln 2} \ln(L_N) - \frac{2}{9} \right] + O(L_N^{-1})\end{aligned}\quad (56)$$

i.e. the energy barrier near zero temperature grows logarithmically with the length $L_N = 2^N$

$$\frac{\ln [2^N t_{eq}^{simple}(L_N = 2^N)]}{\beta} \simeq \frac{4J_0}{3 \ln 2} \ln(L_N) \quad (57)$$

E. Generalization to other dynamics where $G_0(B)$ does not depend explicitly on β

It is clear from Eq. 43 that for all dynamics where $G_0(B)$ does not depend explicitly on β , the equilibrium time will be dominated again by Σ_{min} near zero temperature, with the result (using Eq 40)

$$\begin{aligned}t_{eq}(2^N) &\simeq \frac{e^{\beta \sum_{n=0}^{N-1} J_n}}{2^N} \frac{e^{-\beta \Sigma_{min}}}{G_0(\mathcal{B}_0(\epsilon = -1, \dots, \epsilon_{N-1} = -1))} \\ &\simeq \frac{e^{\beta \sum_{n=0}^{N-1} J_n}}{2^N} \frac{e^{-\beta \Sigma_{min}}}{G_0\left(J_0 + \sum_{m=1}^{N-1} J_m \left(-\frac{1}{2}\right)^m\right)}\end{aligned}\quad (58)$$

i.e. the dynamical barriers are the same as above in Eqs 55 and Eq. 57 (the only difference is in the prefactor of the exponential).

V. EQUILIBRIUM TIME FOR THE GLAUBER DYNAMICS

A. Real Space Renormalization Solution

For the initial condition of Eq. 23 describing the Glauber dynamics, Eq 43 becomes

$$t_{eq}^{Glauber}(2^N) = \frac{1}{2G_N^{final}} = \frac{e^{\beta \sum_{n=0}^{N-1} J_n}}{2^N} \sum_{\epsilon_{N-1}=\pm} \sum_{\epsilon_{N-2}=\pm} \dots \sum_{\epsilon_1=\pm} e^{-\beta \Sigma(\epsilon_1, \dots, \epsilon_{N-1})} \left[e^{\beta \mathcal{B}_0(\epsilon_1, \dots, \epsilon_{N-1})} + e^{-\beta \mathcal{B}_0(\epsilon_1, \dots, \epsilon_{N-1})} \right] \quad (59)$$

B. Leading term near zero temperature

So at low temperature, we have to minimize over the $\epsilon_i = \pm 1$ and over $\epsilon = \pm 1$ the combination (Eqs 45 and Eq 40)

$$\begin{aligned}
\Sigma(\epsilon_1, \dots, \epsilon_{N-1}) + \epsilon \mathcal{B}_0(\epsilon_1, \dots, \epsilon_{N-1}) &= \sum_{m=1}^{N-1} J_m \left[\sum_{n=1}^m \prod_{k=n}^m \left(\frac{\epsilon_k}{2} \right) \right] + \epsilon \left[J_0 + \sum_{m=1}^{N-1} J_m \prod_{k=1}^m \left(\frac{\epsilon_k}{2} \right) \right] \\
&= J_0 \epsilon + J_1 \frac{\epsilon_1}{2} (1 + \epsilon) + J_2 \frac{\epsilon_2}{2} \left[1 + \frac{\epsilon_1}{2} (1 + \epsilon) \right] + J_3 \frac{\epsilon_3}{2} \left[1 + \frac{\epsilon_2}{2} + \frac{\epsilon_1 \epsilon_2}{2^2} (1 + \epsilon) \right] + \dots \\
&+ J_{N-2} \frac{\epsilon_{N-2}}{2} \left[1 + \frac{\epsilon_{N-3}}{2} + \frac{\epsilon_{N-3} \epsilon_{N-4}}{2^2} + \dots + \frac{\epsilon_{N-3} \epsilon_{N-4} \dots \epsilon_1}{2^{N-3}} (1 + \epsilon) \right] \\
&+ J_{N-1} \frac{\epsilon_{N-1}}{2} \left[1 + \frac{\epsilon_{N-2}}{2} + \frac{\epsilon_{N-2} \epsilon_{N-3}}{2^2} + \dots + \frac{\epsilon_{N-2} \epsilon_{N-3} \dots \epsilon_1}{2^{N-2}} (1 + \epsilon) \right] \quad (60)
\end{aligned}$$

The new factors containing ϵ are not able to change the signs of the expressions between brackets, so we find again by recurrence (Eqs 46 and 49) that this function is minimum by choosing all $\epsilon_i = -1$ and finally $\epsilon = -1$. The corresponding minimum reads

$$\begin{aligned}
[\Sigma(\epsilon_1, \dots, \epsilon_{N-1}) + \epsilon \mathcal{B}_0(\epsilon_1, \dots, \epsilon_{N-1})]_{min} &= \Sigma(\epsilon_1 = -1, \dots, \epsilon_{N-1} = -1) - \mathcal{B}_0(\epsilon_1 = -1, \dots, \epsilon_{N-1} = -1) \\
&= \sum_{m=1}^{N-1} J_m \left[\sum_{n=1}^m \left(-\frac{1}{2} \right)^{m-n+1} \right] - \left[J_0 + \sum_{m=1}^{N-1} J_m \left(-\frac{1}{2} \right)^m \right] \\
&= -J_0 + \sum_{m=1}^{N-1} J_m \left[\sum_{k=1}^{m-1} \left(-\frac{1}{2} \right)^k \right] \\
&= -J_0 - \frac{1}{3} \sum_{m=1}^{N-1} J_m \left[1 - \left(-\frac{1}{2} \right)^{m-1} \right] \quad (61)
\end{aligned}$$

We thus obtain the following leading behavior at low temperature (Eq 59)

$$\begin{aligned}
t_{eq}^{Glauber}(2^N) &\simeq \frac{1}{2^N} e^{\beta [\sum_{m=0}^{N-1} J_m - [\Sigma(\epsilon_1, \dots, \epsilon_{N-1}) + \epsilon \mathcal{B}_0(\epsilon_1, \dots, \epsilon_{N-1})]_{min}]} \\
&= \frac{1}{2^N} e^{\beta [\sum_{m=0}^{N-1} J_m + J_0 + \frac{1}{3} \sum_{m=1}^{N-1} J_m [1 - (-\frac{1}{2})^{m-1}]]} \\
&= \frac{1}{2^N} e^{\beta [2J_0 + \sum_{m=1}^{N-1} J_m (\frac{4}{3} - \frac{1}{3} (-\frac{1}{2})^{m-1})]} \quad (62)
\end{aligned}$$

We may now use Eq 2 to compute explicitly

$$\begin{aligned}
\ln [2^N t_{eq}^{Glauber}(2^N)] &= \beta \left[2J_0 + \sum_{m=1}^{N-1} J_m 2^{(1-\sigma)m} \left(\frac{4}{3} - \frac{1}{3} \left(-\frac{1}{2} \right)^{m-1} \right) \right] \\
&= \beta J_0 \left[2 + \frac{4}{3} \sum_{m=1}^{N-1} 2^{(1-\sigma)m} - \frac{1}{3} \sum_{m=1}^{N-1} 2^{(1-\sigma)m} \left(-\frac{1}{2} \right)^{m-1} \right] \\
&= \beta J_0 \left[2 + \frac{4}{3} 2^{1-\sigma} \frac{1 - 2^{(1-\sigma)(N-1)}}{1 - 2^{1-\sigma}} - \frac{1}{3} 2^{1-\sigma} \frac{1 - (-2^{-\sigma})^{(N-1)}}{1 + 2^{-\sigma}} \right] \quad (63)
\end{aligned}$$

This expression is valid for finite N and we may check the first terms

$$\begin{aligned}
\ln [2 t_{eq}^{Glauber}(2)] &= 2\beta J_0 \\
\ln [2^2 t_{eq}^{Glauber}(2^2)] &= \beta J_0 [2 + 2^{1-\sigma}] = \beta [2J_0 + J_1] \quad (64)
\end{aligned}$$

To obtain the leading behavior for large N , we should now specify the value of σ .

C. Case $\sigma < 1$: Power-law barrier

For $\sigma < 1$, Eq 63 becomes in terms of the system size $L_N = 2^N$

$$\begin{aligned} \sigma < 1 : \quad \ln [2^N t_{eq}^{Glauber}(2^N)] &\simeq \beta J_0 \left[\frac{4}{3} \frac{2^{(1-\sigma)N}}{2^{1-\sigma} - 1} + 2 + \frac{4}{3} \frac{1}{2^{\sigma-1} - 1} - \frac{2}{3} \frac{1}{2^\sigma + 1} \right] \\ &= \beta J_0 \left[\frac{4}{3} \frac{L_N^{(1-\sigma)}}{2^{1-\sigma} - 1} + 2 + \frac{4}{3} \frac{1}{2^{\sigma-1} - 1} - \frac{2}{3} \frac{1}{2^\sigma + 1} \right] \end{aligned} \quad (65)$$

i.e. the energy barrier near zero temperature scales as the same power-law of the length $L_N = 2^N$

$$\frac{\ln [2^N t_{eq}^{Glauber}(L_N = 2^N)]}{\beta} \simeq \frac{4J_0}{3(2^{1-\sigma} - 1)} L_N^{(1-\sigma)} \quad (66)$$

and with the same prefactor as in Eq. 55, even if the finite corrections of Eq 65 are different the finite corrections of Eq 54 (more explanations on these finite differences are given in Appendix A).

D. Case $\sigma = 1$: Logarithmic barrier

For $\sigma = 1$, Eq 63 becomes in terms of the system size $L_N = 2^N$

$$\begin{aligned} \sigma = 1 : \quad \ln [2^N t_{eq}^{Glauber}(2^N)] &\simeq \beta J_0 \left[2 + \frac{4}{3}(N - 1) - \frac{2}{9} \right] = \beta J_0 \left[\frac{4}{3}N + \frac{4}{9} \right] \\ &= \beta J_0 \left[\frac{4}{3 \ln 2} \ln L_N + \frac{4}{9} \right] \end{aligned} \quad (67)$$

i.e. the energy barrier near zero temperature grows with the same logarithmic barrier

$$\frac{\ln [2^N t_{eq}^{Glauber}(L_N = 2^N)]}{\beta} \simeq \frac{4J_0}{3 \ln 2} \ln(L_N) \quad (68)$$

with the same prefactor as in Eq 57, even if the finite corrections of Eq. 56 and of Eq. 67 are different (see more details in Appendix A).

E. Generalization to other dynamics where $G_0(B) \propto e^{-\beta|B|}$

From the analysis presented above, it is clear that dynamical barriers near zero temperature will remain the same for all dynamics where the amplitude $G_0(B)$ displays the same exponential decay as the Glauber amplitude of Eq. 23

$$G_0(B) \propto e^{-\beta|B|} \quad (69)$$

As an example of other dynamics with the same exponential decay, we may cite the Metropolis dynamics corresponding to (see Eq 20)

$$G_0^{Metropolis}(B) = \min(e^{\beta B}, e^{-\beta B}) = e^{-\beta|B|} \quad (70)$$

VI. CONCLUSION

In this paper, we have studied the stochastic dynamics of the Dyson hierarchical one-dimensional Ising model of parameter $0 < \sigma \leq 1$ via the Real Space Renormalization introduced in our previous work [25]. We have shown that this renormalization procedure amounts to renormalize a single function $G(B)$ that defines the transition rates of the renormalized dynamics. We have solved explicitly the RG flow for two types of dynamics, namely the 'simple' dynamics (and other equivalent dynamics of section IVE) and the Glauber dynamics (and other equivalent dynamics of section VE). We have obtained that the leading diverging dynamical barrier is the same power-law $\ln t_{eq}(L) \simeq \beta \left(\frac{4J_0}{3(2^{1-\sigma}-1)} \right) L^{1-\sigma}$ for $\sigma < 1$ and the same logarithmic term : $\ln t_{eq}(L) \simeq [\beta \left(\frac{4J_0}{3 \ln 2} \right) - 1] \ln L$ for $\sigma = 1$, even if finite corrections are different, as explained in Appendix A.

Appendix A: Link between dynamical barriers and the highest energy cost of a domain-wall

1. Case of the Glauber dynamics

For the Glauber dynamics, the dynamical barrier for a system of 2^N spins corresponds to the maximal energy cost of one domain wall inside the system

$$\frac{1}{\beta} \ln [2^N t_{eq}^{Glauber}(2^N)] = \max_{0 \leq k \leq 2^N} (U_{2^N}^{(k, 2^N-k)} - U_{2^N}^{GS}) \quad (A1)$$

where $U_{2^N}^{(k, 2^N-k)}$ represents the energy of the configuration where the first k spins are (-1) , whereas all others spins are $(+1)$.

The property of Eq. A1 can be easily checked for small systems, for instance for the case $N = 1$ containing $2^1 = 2$ and the case $N = 2$ containing $2^2 = 4$ spins (Eq 64)

$$\begin{aligned} \frac{1}{\beta} \ln [2 t_{eq}^{Glauber}(2)] &= 2J_0 = U_2^{(1,1)} - U_2^{GS} \\ \frac{1}{\beta} \ln [2^2 t_{eq}^{Glauber}(2^2)] &= 2J_0 + J_1 = U_4^{(1,3)} - U_4^{GS} = U_4^{(3,1)} - U_4^{GS} \end{aligned} \quad (A2)$$

For an arbitrary number N of generations containing 2^N spins, the correspondence of Eq. A1 is less straightforward because the energy cost of the configuration where the first k spins are (-1) , whereas all others spins are $(+1)$ reads

$$U_{2^N}^{(k, 2^N-k)} - U_{2^N}^{GS} = 2J_0 c_0(k) + \sum_{j=1}^{N-1} J_j [2c_j(k) + (1 - 2c_j(k)) 2^{1-j} \Sigma_{j-1}(k)] \quad (A3)$$

in terms of the coefficients $c_i(k) \in \{0, 1\}$ of the base-two decomposition

$$k = \sum_{i=0}^{N-1} c_i(k) 2^i = c_0(k) + c_1(k) 2 + c_2(k) 2^2 + \dots \quad (A4)$$

and of the corresponding partial sums for $i = 0, \dots, N-1$

$$\Sigma_i(k) = \sum_{j=0}^i c_j(k) 2^j \quad (A5)$$

Nevertheless, one can check that the energy barrier obtained in Eq. 62 corresponds to the energy cost of a domain wall located at k_*

$$\frac{1}{\beta} \ln (2^N t_{eq}^{Glauber}(2^N)) \simeq 2J_0 + \sum_{m=1}^{N-1} J_m \left[\frac{4}{3} - \frac{1}{3} \left(-\frac{1}{2} \right)^{m-1} \right] = (U_{2^N}^{(k_*, 2^N-k_*)} - U_{2^N}^{GS}) \quad (A6)$$

where k_* is characterized by the following coefficients in the base-two decomposition A4

$$\begin{aligned} c_{2p}(k_*) &= 1 \\ c_{2p+1}(k_*) &= 0 \end{aligned} \quad (A7)$$

so that

$$k_* = 1 + 2^2 + 2^4 + \dots \quad (A8)$$

2. Case of the simple dynamics

For the 'simple' dynamics, the correspondence of Eq. A1 between the dynamical barrier and the maximal energy cost of a single domain wall does not hold, as can be seen already for the case $N = 2$ corresponding to $2^1 = 2$ spins

and for the case $N = 2$ corresponding to $2^2 = 4$ spins, since the dynamical barriers of Eq 53

$$\begin{aligned}\frac{1}{\beta} \ln [2t_{eq}^{simple}(2)] &= J_0 \\ \frac{1}{\beta} \ln [2^2 t_{eq}^{simple}(2^2)] &= J_0 + \frac{3}{2} J_1\end{aligned}\tag{A9}$$

are clearly different from the maximal energy cost of Eq A2. This can be understood as follows on the case $N = 1$ with two spins. The transitions rates associated to the simple dynamics (Eqs 20 and 22) read

$$\begin{aligned}W^{simple}(++ \rightarrow +-) &= W^{simple}(++ \rightarrow -+) = e^{-\beta J_0} \\ W^{simple}(+- \rightarrow ++) &= W^{simple}(+- \rightarrow --) = e^{+\beta J_0}\end{aligned}\tag{A10}$$

whereas the Glauber transition rates (Eq 23) are given by

$$\begin{aligned}W^{Glauber}(++ \rightarrow +-) &= W^{Glauber}(++ \rightarrow -+) = \frac{e^{-\beta J_0}}{e^{+\beta J_0} + e^{-\beta J_0}} = \frac{e^{-2\beta J_0}}{1 + e^{-2\beta J_0}} \\ W^{Glauber}(+- \rightarrow ++) &= W^{Glauber}(+- \rightarrow --) = \frac{e^{+\beta J_0}}{e^{+\beta J_0} + e^{-\beta J_0}} = \frac{1}{1 + e^{-2\beta J_0}}\end{aligned}\tag{A11}$$

For 2 spins, the equilibrium time is determined by the rate $W(++ \rightarrow + -)$ to create a domain-wall when starting from one ground state (the time to eliminate the domain-wall is then negligible), and these two rates are respectively of order $e^{-\beta J_0}$ for the simple dynamics and of order $e^{-2\beta J_0}$ for the Glauber dynamics, i.e. the dynamical barriers differ by a factor 2. One could argue that the Glauber dynamics (or other equivalent dynamics of section V E) is more 'physical', in the sense that all transitions rates remain bounded near zero-temperature, whereas in the 'simple' dynamics (or other equivalent dynamics of section IV E), transition rates corresponding to a decrease of the energy diverge near zero temperature. Nevertheless, we should stress that the difference between the dynamical barriers of the two dynamics remains of order $O(1)$, whereas the leading terms found in the text for the case $\sigma < 1$ (Eqs 55 and 66) and for the case $\sigma = 1$ (Eqs 57 and 68) are the same.

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